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MODELING OF SHEAR LOCALIZATION

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The process of shear localization on discrete systems of lines is considered. The shear lines are modeled by curved mathematical cuts with conditions on them providing the possibility of appearing of discontinuities in shear displacements. A class of stress functions is constructed, which allow one to describe the stress-strain state in an elastic ring with an arbitrary number of shear cracks in the form of logarithmic spirals.

Key words: *shear localization, shear cracks, stress function.*

Formulation of Problems with Discrete Systems of Shear Lines. Shear localization on individual surfaces can be observed in rocks, loose media, and metals [1–3]. In the present work, we simulate the process of shear localization on systems of lines in a plane domain Ω bounded by the curve Γ_Ω. Within the framework of the method of successive loading, we assume that the number of shear lines, their shape, and positions in the domain are determined at a given loading step. The shear lines are represented in the form of mathematical cuts. The edges of the cuts Γ^+ and Γ^- are considered as parts of the common boundary $\Gamma = \Gamma_\Omega \cup \Gamma^+ \cup \Gamma^-$ of the examined domain.

We pose the problem of finding the fields of displacement increments u_i $(i = 1, 2)$ and stress increments σ_{ij} $(i, j = 1, 2)$ at each loading step in the domain Ω with the boundary Γ. The sought functions should satisfy the equilibrium equations

$$\sigma_{ij,j} = X_i,$$

where X_i are the components of the vector of increments of body forces in the Cartesian coordinate system.

To identify effects associated with shear localization, we choose the model of a linearly elastic body for the material outside the lines. In the case of plane deformation, Hooke's law has the form [4]

$$\sigma_{ij} = \frac{E}{1+\nu}\Big(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\varepsilon_{kk}\Big),$$

where ε_{ij} $(i, j = 1, 2)$ are the components of the strain-rate tensor

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2;$$

E is Young's modulus, and ν is Poisson's ratio; summation over k is performed from 1 to 2.

We present the equations of state in the shear-localization region in the form of boundary conditions that describe interaction of the cut edges $\Gamma^\pm$. It follows from conservation laws [5] that the normal component of the vector of displacement increments u_n and the vector of displacement increments are continuous on the area of the tangential discontinuity of displacements:

$$\Gamma^\pm: \qquad u_n^+ = u_n^-; \tag{1}$$

$$\Gamma^\pm: \qquad (p_n^n)^+ = (p_n^n)^-, \qquad (p_\tau^n)^+ = (p_\tau^n)^-. \tag{2}$$

Here, the subscripts plus and minus refer to different sides of the discontinuity line; p_n^n and p_τ^n are the normal and tangential components of the vector of stress increments. The direction cosines of the normal and tangent lines are chosen identical on different sides of the cut.

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Modeling of shear-localization lines by cuts provides a possibility of sliding of the cut edges with respect to each other, i.e., displacement of the initially coinciding points along certain lines. Such a possibility is realized, depending on conditions specified on the cuts and their relation to loading conditions and equations of state.

Origination of a tangential discontinuity of displacements at a given point on the shear line is possible if one of the following conditions on different parts of the cuts is satisfied:

$$\Gamma_1^{\pm}: \qquad p_\tau^n = g_1(p_n^n, x_1, x_2); \tag{3}$$

$$\Gamma_2^{\pm}: \qquad g_2(u_\tau^+ - u_\tau^-, p_\tau^n, x_1, x_2) = 0. \tag{4}$$

Here u_τ is the tangential component of the vector of displacement increments, x_1 and x_2 are Cartesian coordinates of the point, and g_1 and g_2 are given functions.

On the parts of the cuts $\Gamma_c^{\pm}$ with no discontinuities arising along them, in addition to conditions (1) and (2), condition (4) with the function $g_2 \equiv u_\tau^+ - u_\tau^-$ should be satisfied.

The function g_2 in (4) can be written in the form of an explicit dependence of the increment of shear stress p_τ^n on the jump of the increment of shear displacement $u_\tau^+ - u_\tau^-$ and, in particular, can describe softening of a material.

Thus, we present the system of cuts in the domain under study in the form

$$\Gamma^{\pm} = \Gamma_1^{\pm} \cup \Gamma_2^{\pm} \cup \Gamma_c^{\pm}.$$

The boundary conditions (1), (2), and (3) or (4) are satisfied on two edges of the cut $\Gamma^{\pm}$.

The shape and position of the shear-localization lines are given or are determined (refined) in the course of solving the problem on the basis of certain criteria. For example, as a criterion, we consider the condition

$$\tau + \sigma \sin\phi - \tau^* \cos\phi < 0,$$

where $\tau = \sqrt{(\sigma_{11}^l - \sigma_{22}^l)^2 + 4(\sigma_{12}^l)^2}/2$, $\sigma = (\sigma_{11}^l + \sigma_{22}^l)/2 < 0$ [σ_{ij}^l $(i, j = 1, 2)$ are total stresses at the current lth step of loading], and τ^* and ϕ are given parameters of the material. If this condition is violated in the vicinity of the expected shear line, we set the Coulomb friction law on it: condition (3) with the function $g_1 = \pm(\tau^l - p_n^n \tan\phi)$, where the choice of the sign depends on the normal direction and loading history and the value of τ^l depends on the loading history and τ^*.

In turn, the boundary Γ_Ω can contain parts with conditions of different types, which relate the normal and tangential components of the vector of displacement increments and the corresponding components of the vector of stress increments on these parts of the boundary.

The algorithm of numerical solution of the posed problem is constructed on the basis of the finite-element method [6]. The domain is initially divided into finite elements automatically, with allowance for available experimental data and analytical studies of positions of localization lines for each problem. One family of lines of the finite-element grid is chosen to be maximally close to the family of localization lines in shape and direction. The grid can be corrected during solving the problem, depending on the criterion of propagation of lines and conditions on them.

An important feature of the initial finite-element grid is that all its nodes are double nodes. This allows us to located the cuts not only along any family but simultaneously along several families of the grid. Numeration of grid nodes is optimized to reduce the volume of data structures used in the program; the positions of the cuts are taken into account. The distances between the cuts are limited from below only by the element size. Therefore, the number of shear lines and the distance between them in the system can be arbitrary.

To implement nonstandard boundary conditions within the finite-element method, we use the property of linearity of the system of finite elements at each loading step. The algorithm developed allows one to solve problems with functional dependences between unknown stresses and displacements being set at the boundary.

Along with simplex elements, we consider cubic Hermite finite elements. In nodes of Hermite finite elements, the sought function is the vector function of displacement increments and their derivatives. This allows one to increase the accuracy of calculating the stress-tensor components at the points of the computational grid under conditions of shear-line propagation. The system of equations of the finite-element method is derived on the basis of the minimum potential energy principle.

For the numerical solution, a program package is developed, which includes programs of generation of problem-oriented grids with double nodes, construction of stiff matrices for various models and implementation of conditions on cuts, by virtue of which discontinuities can appear inside a continuous region, and also programs for obtaining fields of displacements and stresses and visualization of the deformation pattern.

Solutions of particular problems on material deformation under conditions of shear localization on systems of logarithmical spirals in the vicinity of a circular orifice [7], on systems of Cassinian ovals [8], on systems of straight lines in the case of simple shear [9], and other problems were obtained.

Analytical Study of the Problem of Shear Cracks in a Ring. Let us consider an elastic ring ($1 \leqslant r \leqslant R$) with n shear cracks represented as cuts located along the curves $\xi = (2m-1)\alpha$ ($\alpha = \pi/n$, $m = 1, \ldots, n$, $n \in N$, and $R = \text{const}$). We choose these curves as logarithmic spirals $\xi = \theta - \mu \ln r$ intersecting the radii at an angle $\pi/4 + \phi/2$, where $\mu = \tan\,(\pi/4 + \phi/2)$, $\phi = \text{const}$ ($0 \leqslant \phi < \pi/2$), and r and θ are the polar coordinates. Along with the numerical analysis, the analytical study of this problem is also of interest.

The linearly elastic behavior of the material in the absence of body forces in the plane case allows us to express the stresses in terms of the biharmonic function of stresses [4].

For shear cracks on the cut edges, the conditions of continuity of the normal displacement and stress vector acting on the tangential area should be satisfied. Thus, we pose the following problem: find a stress-strain state in the domain considered, such that the normal displacement and stress vector were continuous periodic functions of the variable ξ with a period 2α. In this case, it is sufficient to obtain the stress-strain state of the elastic material in the region bounded by the curves $\xi = \pm\alpha$, $r = 1$, $r = R$, and then continue it periodically to the whole ring.

We pose the problem of finding, in the domain considered, the biharmonic stress functions in the class of functions of the form

$$U_k = r^{2k+2}\Phi_k(\xi).$$

Substituting the sought functions into the biharmonic equation

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}\Big)\Big(\frac{\partial^2 U_k}{\partial r^2} + \frac{1}{r}\,\frac{\partial U_k}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 U_k}{\partial \theta^2}\Big) = 0,$$

for the functions $\Phi_k(\xi)$, we obtain the fourth-order ordinary differential equation

$$(\mu^2+1)^2\frac{d^4\Phi_k}{d\xi^4} - 4\mu(\mu^2+1)^2(2k+1)\,\frac{d^3\Phi_k}{d\xi^3} + 4(\mu^2(6k^2+6k+1) + 2k^2+2k+1)\frac{d^2\Phi_k}{d\xi^2}$$

$$- 16\mu k(2k+1)(k+1)\,\frac{d\Phi_k}{d\xi} + 16k^2(k+1)^2\Phi_k = 0.$$

The fundamental systems of solutions of this equation, depending on k, have the following form:

— for $k = -1$,

$$1, \qquad \xi, \qquad \exp\Big(-\frac{2(\mu+i)\xi}{\mu^2+1}\Big), \qquad \exp\Big(-\frac{2(\mu-i)\xi}{\mu^2+1}\Big);$$

— for $k = 0$,

$$1, \qquad \xi, \qquad \exp\Big(\frac{2(\mu+i)\xi}{\mu^2+1}\Big), \qquad \exp\Big(\frac{2(\mu-i)\xi}{\mu^2+1}\Big);$$

— for $k \neq -1, 0$

$$\exp\Big(\frac{2k(\mu+i)\xi}{\mu^2+1}\Big), \qquad \exp\Big(\frac{2k(\mu-i)\xi}{\mu^2+1}\Big),$$

$$\exp\Big(\frac{2(k+1)(\mu+i)\xi}{\mu^2+1}\Big), \qquad \exp\Big(\frac{2(k+1)(\mu-i)\xi}{\mu^2+1}\Big).$$

We present the stress function in terms of the complex potentials $\varphi_k(z)$ and $\chi_k(z)$ by Goursat's formula [4]

$$U_k = (\bar z\varphi_k(z) + z\overline{\varphi_k(z)} + \chi_k(z) + \overline{\chi_k(z)})/2, \tag{5}$$

where $z = x + iy = r\,\mathrm{e}^{i\theta}$ and $\bar z = x - iy = r\,\mathrm{e}^{-i\theta}$.

We choose the complex potentials in the form

$$\varphi_{-1}(z) = (a_{-1} + ib_{-1})z^{\frac{2(\mu i-1)}{\mu^2+1}+1}, \quad \chi_{-1}(z) = (c_{-1} + id_{-1})(1 + (\mu+i)\ln z) \quad (k = -1),$$

$$\varphi_0(z) = (a_0 + ib_0)z(1 + (\mu+i)\ln z), \qquad \chi_0(z) = (c_0 + id_0)z^{\frac{2(1-\mu i)}{\mu^2+1}+1} \qquad (k = 0),$$

$$\varphi_k(z) = (a_k + ib_k)z^{\frac{2k(1-\mu i)}{\mu^2+1}+1}, \qquad \chi_k(z) = (c_k + id_k)z^{\frac{2(k-1)(1-\mu i)}{\mu^2+1}+1} \qquad (k \neq -1, 0),$$

where a_k, b_k, c_k, and d_k are arbitrary constants.

For $\phi = 0$, the logarithmic spirals along which the cracks are located make an angle $\pi/4$ with the radius; hence, $\mu = 1$. For this case, the stresses and displacements in polar coordinates for each function U_k (5) are found by the formulas [4]

$$\sigma_r + \sigma_\theta = 2[\varphi_k'(z) + \overline{\varphi_k'(z)}], \qquad \sigma_\theta - \sigma_r + 2i\sigma_{r\theta} = 2\,\mathrm{e}^{2i\theta}[\bar{z}\varphi_k''(z) + \chi_k''(z)],$$

$$2G(u_r + iu_\theta) = \mathrm{e}^{-i\theta}[æ\varphi_k(z) - z\overline{\varphi_k'(z)} - \overline{\chi_k'(z)}],$$

where $G = E/(2(1+\nu))$ and $æ = 3 - 4\nu$.

For $k \neq -1, 0$, we obtain the components of the stress tensor in the form

$$\sigma_r = r^{2k}\,\mathrm{e}^{k\xi}\{[a_k(2+k) + b_k k(1-2k)]\cos k\xi + [a_k k(1-2k) - b_k(2+k)]\sin k\xi\}$$
$$+ r^{2k}\,\mathrm{e}^{(k+1)\xi}(k+1)\{[c_k - d_k(1+2k)]\cos(k+1)\xi - [c_k(1+2k) + d_k]\sin(k+1)\xi\},$$
$$\sigma_\theta = r^{2k}\,\mathrm{e}^{k\xi}\{[a_k(2+3k) + b_k k(3+2k)]\cos k\xi + [a_k k(3+2k) - b_k(2+3k)]\sin k\xi\}$$
$$+ r^{2k}\,\mathrm{e}^{(k+1)\xi}(k+1)\{[d_k(1+2k) - c_k]\cos(k+1)\xi + [c_k(1+2k) + d_k]\sin(k+1)\xi\},$$
$$\sigma_{r\theta} = r^{2k}\,\mathrm{e}^{k\xi}\,k\{[b_k - a_k(1+2k)]\cos k\xi + [a_k + b_k(1+2k)]\sin k\xi\}$$
$$- r^{2k}\,\mathrm{e}^{(k+1)\xi}(k+1)\{[c_k(1+2k) + d_k]\cos(k+1)\xi + [c_k - d_k(1+2k)]\sin(k+1)\xi\},$$

and the components of the displacement vector in the form

$$u_r = r^{2k+1}\,\mathrm{e}^{k\xi}\{[a_k(æ - k - 1) - b_k k]\cos k\xi + [b_k(k - æ + 1) - a_k k]\sin k\xi\}/(2G)$$
$$- r^{2k+1}\,\mathrm{e}^{(k+1)\xi}(k+1)[(c_k + d_k)\cos(k+1)\xi + (c_k - d_k)\sin(k+1)\xi]/(2G),$$
$$u_\theta = r^{2k+1}\,\mathrm{e}^{k\xi}\{[b_k(æ + k + 1) - a_k k]\cos k\xi + [a_k(æ + k + 1) + b_k k]\sin k\xi\}/(2G)$$
$$+ r^{2k+1}\,\mathrm{e}^{(k+1)\xi}(k+1)[(d_k - c_k)\cos(k+1)\xi + (c_k + d_k)\sin(k+1)\xi]/(2G).$$

We consider the conditions of equality of the stress vector and the normal component of the displacement vector on the lines $\xi = \pm\alpha$ for arbitrary fixed r. We assume that, for each k, the difference between the values of shear displacement for $\xi = \pm\alpha$ is $r^{2k+1}\sqrt{2}(æ+1)V_k/G$. Solving the resultant system of four equations, we find

$$a_k = V_k\,\frac{\cos k\alpha \sinh k\alpha + \sin k\alpha \cosh k\alpha}{\cos^2 k\alpha \sinh^2 k\alpha + \sin^2 k\alpha \cosh^2 k\alpha}, \qquad b_k = V_k\,\frac{\cos k\alpha \sinh k\alpha - \sin k\alpha \cosh k\alpha}{\cos^2 k\alpha \sinh^2 k\alpha + \sin^2 k\alpha \cosh^2 k\alpha},$$

$$c_k = -V_k\,\frac{\cos(k+1)\alpha \sinh(k+1)\alpha + \sin(k+1)\alpha \cosh(k+1)\alpha}{\cos^2(k+1)\alpha \sinh^2(k+1)\alpha + \sin^2(k+1)\alpha \cosh^2(k+1)\alpha},$$

$$d_k = -V_k\,\frac{\cos(k+1)\alpha \sinh(k+1)\alpha - \sin(k+1)\alpha \cosh(k+1)\alpha}{\cos^2(k+1)\alpha \sinh^2(k+1)\alpha + \sin^2(k+1)\alpha \cosh^2(k+1)\alpha}.$$

Thus, four conditions on two boundaries $\xi = \pm\alpha$ determine, for each $k \neq -1, 0$, all constants in the function U_k.

By analogy with the classical problem of bending of a circular beam [4], we confine ourselves to setting the main vector and main moment of forces on the remaining parts of the boundary (circular arcs). According to the St. Venant principle, such definition of the boundary conditions is allowed if R is sufficiently large and α is comparatively small.

In the case $k = 0$, the stress-tensor components take the form

$$\sigma_r = -2(a_0 + b_0)\theta + (a_0 - b_0)\ln r + 3a_0 - b_0 + \mathrm{e}^{\xi}[(c_0 - d_0)\cos\xi - (c_0 + d_0)\sin\xi],$$
$$\sigma_\theta = -2(a_0 + b_0)\theta + (a_0 - b_0)\ln r + 5a_0 - 3b_0 + \mathrm{e}^{\xi}[(d_0 - c_0)\cos\xi + (c_0 + d_0)\sin\xi],$$
$$\sigma_{r\theta} = -\,\mathrm{e}^{\xi}[(c_0 + d_0)\cos\xi + (c_0 - d_0)\sin\xi] + a_0 + b_0,$$

and the components of the displacement vector take the form

$$u_r = r\{(1 - æ)[(a_0 + b_0)\theta + (b_0 - a_0)\ln r - a_0] + b_0 - a_0\}/(2G)$$
$$- r\,\mathrm{e}^{\xi}[(c_0 + d_0)\cos\xi - (d_0 - c_0)\sin\xi]/(2G),$$
$$u_\theta = r\{(1 + æ)[(a_0 - b_0)\theta + (b_0 + a_0)\ln r + b_0] + b_0 + a_0\}/(2G)$$
$$+ r\,\mathrm{e}^{\xi}[(d_0 - c_0)\cos\xi + (c_0 + d_0)\sin\xi]/(2G).$$

In the case $k = -1$, we obtain the stress-vector components

$$\sigma_r = r^{-2}\{\mathrm{e}^{-\xi}[(a_{-1} - 3b_{-1})\cos\xi + (3a_{-1} + b_{-1})\sin\xi] + c_{-1} - d_{-1}\},$$

$$\sigma_\theta = -r^{-2}\{\mathrm{e}^{-\xi}[(a_{-1} + b_{-1})\cos\xi - (a_{-1} - b_{-1})\sin\xi] + c_{-1} - d_{-1}\},$$

$$\sigma_{r\theta} = -r^{-2}\{\mathrm{e}^{-\xi}[(b_{-1} + a_{-1})\cos\xi - (a_{-1} - b_{-1})\sin\xi] + c_{-1} + d_{-1}\}$$

and the displacement-vector components

$$u_r = \mathrm{e}^{-\xi}[(a_{-1}æ + b_{-1})\cos\xi + (b_{-1}æ - a_{-1})\sin\xi]/(2Gr) - (c_{-1} - d_{-1})/(2Gr),$$

$$u_\theta = \mathrm{e}^{-\xi}[(b_{-1}æ + a_{-1})\cos\xi - (a_{-1}æ - b_{-1})\sin\xi]/(2Gr) - (c_{-1} + d_{-1})/(2Gr).$$

In each of these two cases, it is also possible to set the difference in shear displacements and equality of the corresponding components of the stress vector and normal components of the displacement vector on the lines $\xi = \pm\alpha$ for arbitrary r.

Thus, if the function of discontinuity of shear displacements on cracks is specified in the form

$$\frac{\sqrt{2}(æ + 1)}{G} \sum_k V_k r^{2k+1}$$

the stress function for the problem considered can be set as $\sum\limits_k U_k$.

Instead of the condition of a given discontinuity of shear displacements on the lines $\xi = \pm\alpha$, we can consider conditions of the type (3) or (4). It was assumed [10] that sliding with a constant shear stress τ^* occurs along the cracks. A solution was obtained for the stress function constructed with the use of U_0 and U_{-1}, which coincides with the classical ideally plastic solution [1] as $\alpha \to 0$ under the condition that the maximum shear stress equals a constant value τ^* within the entire deformation domain. The shear cracks in this limiting case have the meaning of slip lines for the continuum model.

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